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(July)

MATHEMATICS

(Honours)

(Mathematical Modeling)

(HOPT–62 : OP6)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, taking **one** from each Unit

UNIT—I

1. *(a)* For a two-species model, consider the following system of differential equations :

$$\frac{dx}{dt} = 9x + 4y + 3$$

$$\frac{dx}{dt} = -5x + 3y + 6$$

(i) Is the above system of equations autonomous? Justify your answer.

(ii) Give the conditions when the critical point of a system of differential equations is a node.

(iii) Find the eigenvalues and eigen-vectors of the corresponding homogeneous system of the problem given above. 3+3+3=9

(b) A body is falling free in a vacuum. The fall is necessarily related to the gravitational acceleration g and the height h from which the body is dropped. Use dimensional analysis to show that the velocity V of the falling body satisfies the relation

$$V / \sqrt{gh} = \text{constant}$$ 6

2. *(a)* A particle falls from rest in a medium in which the resistance is γv^2 per unit mass. Prove that the distance fallen in time t is

$$\frac{1}{\gamma}\cosh(t\sqrt{g\lambda})$$ 7

(b) Consider the motion of a simple pendulum. Write the differential equation describing the motion of the simple pendulum with initial conditions

$\theta = \theta_0$ at $t = 0$ and $\frac{d\theta}{dt} = 0$ at $t = 0$, for both the linear and non-linear models. Give the expression for the time period for both the cases. In the linear case, derive the time period. Suppose the length of the pendulum is doubled, then find the time period of the pendulum in both the cases. 8

UNIT—II

3. *(a)* Consider a single-server queueing system with Poisson input and exponential service times. Suppose the mean arrival rate is 3 calling units per hour, the expected service time is 0·25 hour and the maximum permissible calling unit in the system is 2. Derive the steady- state probability distribution of the number of calling units in the system and then calculate the expected number in the system. 6

(b) The growth of a population is proportional to the population and restricted by availability of food, space, etc., which can be modelled as proportional to the square of the population itself.

(i) Model this process.

(ii) Solve the resulting equation.

(iii) Show that the population approaches a limiting value. Give the interpretation of this value in the given context. 3+3+3=9

4. *(a)* Give one example each from the real world of the following with justification for your example : 2+2+2=6

(i) A stochastic model

(ii) A non-linear model

(iii) A queueing model

(b) The mean arrival rate to a service centre is 5 per hour. The mean service time is found to be 1 minute for service. Assuming Poisson arrival and exponential service time, find—

(i) the utilization factor for this service facility;

(ii) the probability of two units in the system;

(iii) the expected number of units in the system;

(iv) the expected time in hour that a customer has to spend in the system. 2+2+2+3=9

UNIT—III

5. *(a)* In a population of lions, the proportionate death rate is 0·55 per year and the proportionate birthrate is 0·45 per year. Formulate a model of the population. Solve the model and discuss its long-term behaviour. Also find the equilibrium point of the model. 7

(b) The rate of increase of susceptible SARS victims (persons who may contract SARS) is proportional to the number of susceptible persons and number of infected persons. If there are no susceptible persons and 1 infected person at a time, set up the equation for the spread of the disease. Solve the resulting equation. 8

6. *(a)* Consider the case of a disease which is lethal to all those contracting it, i.e., all remarks are in fact deaths and make no contribution to the life of the community. New susceptible births arise solely from the susceptible group itself. Formulate this model and examine the steady state. What type of waves do the epidemic cycles consist of? 7

(b) The model for the number of infectives y of a population affected by the spread of a non-fatal disease results in a differential equation

$$\frac{dy}{dt} = y(N\beta - Y - \beta y), \; y(0) = y_0$$

where N the total population, y_0 the initial infected population, y the recovery rate and β the contact rate and all are constants. Solve for y and show that the epidemic converges exponentially to the stable state. 8

UNIT—IV

7. *(a)* In a car garage, cars arrive at a rate of 50 cars per day. Assuming that inter-arrival time follows an exponential distribution and the service time distribution is also exponential with an average of 30 minutes, calculate the following : 6

(i) Average number of cars in the queue

(ii) The probability that the queue size is greater than or equal to 6

(b) Suppose that there are two duopolists whose production costs are respectively

$$C_1 = q_1^2 + 8q_1 + 8 \text{ and } C_2 = 0 \cdot 625q_2^2 + 5q_2 + 5$$

and the demand curve facing them is $q = 200 - 10p$, where q is the total quantity demanded and p is the price assumed to be the same for both the duopolists. Find the levels of outputs that maximize their profit and obtain the equilibrium price. Assume that the conjectural variations are zero, i.e.,

$$\frac{dq_2}{dq_1} = 0 = \frac{dq_1}{dq_2}$$ 9

8. *(a)* The quarterly production of washing machines in a factory for three quarters were 2500, 2625 and 2850 respectively. Use exponential smoothing based upon the first three observations, to forecast production for the fifth period, using $\alpha = 0 \cdot 1$ and $\beta = 0 \cdot 2$, where $\hat{y} = \alpha + \beta x$ gives the best fitted line to the data relationship. From past data (prior to the three data points), a straight line was fit. The value on the line corresponding to the last observed time is 2450 and the slope is 90. 8

(b) Find the output yielding maximum profit 4 for the cost function

$$C = 0 \cdot 7x^3 + 0 \cdot 8x^2 + 14x + 5$$

given that the cost price of x is ₹ 45 per unit. 7

UNIT—V

9. *(a)* Consider arterial blood viscosity $\mu = 0 \cdot 025$ poise. If the length of the artery is 1·5 cm, radius 8×10^{-3} cm and $P = P_1 - P_2 = 4 \times 103$ dynes/cm^2, then find the—

(i) maximum peak velocity of blood;

(ii) shear stress at the wall. 8

(b) A solid tumour usually grows at a declining rate because its interior has no access to oxygen and other necessary substances that circulation supplies. This situation has been modelled by its Gompertz growth law

$$\frac{dN}{dt} = \gamma N$$

where $\frac{d\gamma}{dt} = -\alpha\gamma$, where γ is the effective growth rate, which will decrease exponentially by our assumption. Show that this can be equivalently expressed as

$$\frac{dN}{dt} = \gamma_0 e^{\alpha t}, \ \ N = (-\alpha \ln N)\, N$$ 7

10. *(a)* A drug is induced in patient's bloodstream at a constant rate of r gm/sec. Simultaneously the drug is removed at a rate proportional to the amount $x(t)$ of the drug present at any time t. Determine the differential equation governing the amount $x(t)$. If the initial concentration of the drug in the bloodstream is x_0, find the concentration of the drug at any time t. 7

(b) Consider the blood flow in an artery following Poiseuille's law. If the length of the artery is 3 cm, radius is 7×10^{-3} cm and driving force is 5×10^3 dynes / cm^2, then using blood viscosity, $\mu = 0 \cdot 027$ poise, find the—

(i) velocity $u(y)$ and the maximum peak velocity of blood;

(ii) shear stress at the wall of the artery. 8

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